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The torque on a rotating disk in the surface of a liquid with an adsorbed film

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SUMMARY
In this paper we consider the problem of calculating the resistive torque on a disk rotating slowly with constant angular speed in the surface of a liquid with an adsorbed surface film. Using the method of complementary representations for generalised axially symmetric potential functions, the boundary-value problem for the azimuthal velocity component is reduced to the solution of a Fredholm integral equation of the second kind. This equation is solved numerically and asymptotically for all values of the ratio of the surface shear viscosity of the film to the viscosity of the substrate fluid, and values calculated for the substrate and film torques on the disk. The results are compared with previous work of Goodrich and his co-workers.

1. Introduction

In a recent series of papers [1 to 5] Goodrich and his collaborators have studied the dynamics of a rotating disk viscometer for the measurement of the surface shear viscosity of an adsorbed film. In essence the apparatus consists of a thin circular disk or annulus inserted into the plane interface between a very thin film of viscous fluid and an underlying substrate of different viscous fluid. The disk is rotated slowly, and the torque necessary to maintain the steady rotation is measured. From a knowledge of this torque, the viscosity of the substrate and suitable theoretical formulae, the surface shear viscosity of the film is deduced. The work of Goodrich et al. was directed at providing theoretical information on the driving torque in various configurations.

In [1] an analysis is made of the problem of a rotating circular disk. The fluid motion is assumed to be steady and slow enough for the linearised Stokes equations to be used; the only non-vanishing component of fluid velocity w is then in the azimuthal direction. The problem of determining w is of the mixed boundary-value type, but its mathematical novelty lies in the nature of the condition imposed on the surface of the fluid outside the disk. If the x-axis of a system of cylindrical polar coordinates (ρ, x, ϕ) is drawn into the fluid normal to the surface then, outside the area covered by the rotating disk, the balance of substrate stresses on the adsorbed film and the internal film stresses is expressed by the "generalised impedance condition"

$$\mu \frac{\partial w}{\partial x} - \eta \frac{\partial^2 w}{\partial x^2} = 0,$$

where μ and η are respectively the coefficient of internal viscosity of the substrate and surface shear viscosity of the adsorbed film. (Boundary conditions of a similar structure also appear in geomagnetic problems concerned with currents induced by varying magnetic fields in finitely conducting sheets [6, 7].) Over the disk region the surface fluid velocity must equal that of the disk.

The approach of [1] is to use a Hankel-type representation of w and to formulate the mixed boundary conditions on the fluid surface as a pair of dual integral equations. By various manipulations the solution of the dual relations is reduced to that of a Fredholm integral equation of the second kind for the unknown function in the Hankel representation of w. This integral equation is then reformulated as a variational principle, and an approximate solution found by the optimisation of suitable trial solutions. These results are used in [2] to calculate approximations to the film and substrate contributions to the torque on the disk.

There are several respects in which the analysis of [1] and [2] seems not to be wholly satisfactory. First, the solution of the special case $\mu = 0$ in Section 3 of [1] is seen to contain a divergent integral if the solution (6) of the dual relations (5) is substituted back into $(5)_2$. The surface velocity field is, however, given correctly by (8) of [1], and an alternative elementary derivation is given in Appendix 1 to this paper. Second, the governing Fredholm integral equation (15) of [1] is over an infinite range, and appears to be unsuitable for asymptotic solution as either $\lambda \to 0$ or $\lambda \to \infty$, where $\lambda = \eta/\mu$. The infinite range means that several of the standard numerical techniques of integration are not available, and the variational approach adopted involves much elaborate analysis of integrals of the Weber–Schafheitlin type. Finally, in [2] there is the problem of deciding for small λ which of two film-torque curves is the correct one.

In view of these points and because of the intrinsic mathematical interest of the mixed boundary-value problem involved, a re-examination of the circular disk viscometer problem has been undertaken. In Section 2 of this work the problem is formulated in the language of Generalised Axially Symmetric Potential Theory (GASPT) as a mixed boundary-value problem for a five-dimensional potential function. Using the method of complementary representations, developed by Ranger [8] and Shail [9] and summarised in Appendix 2, the five-dimensional problem, with the generalised impedance condition off the disk, is mapped onto a four-dimensional problem with a conventional impedance condition imposed outside the image of the disk. A representation for the four-dimensional potential function is then established using Green's theorem (a technique not readily available in five dimensions due to the unusual nature of the generalised impedance condition), and a Fredholm integral equation of the first kind derived for an unknown source distribution. This equation is unsuitable for numerical analysis since the source distribution is expected to have an integrable singularity at one end of the (finite) integration range; thus standard methods [10] are used to convert the equation to a Fredholm equation of the second kind for a derived quantity $T(u)$ which is regular over the interval of definition of the integral equation. Expressions are then obtained in terms of $T(u)$ for the substrate and film torques acting on the disk.

In Section 3 the governing integral equation is solved numerically by first reducing it to an infinite set of simultaneous linear equations in the coefficients of a suitable Fourier–Dini expansion of $T(u)$, a procedure suggested by Mangulis [11], who treats an equation of similar structure which arises in the study of sound radiation from a baffled circular piston. The infinite set of equations is truncated and solved on a computer, and the substrate and film torques evaluated. It is found that convergence with respect to $\mathcal{N}$, the size of the truncated system of equations, is satisfactory except for small values of the ratio η/μ, and sample numerical results are given for a range of values of this parameter.

In Section 4, asymptotic expressions are developed for the substrate and film torques in both of the limits $\lambda \to \infty$ and $\lambda \to 0$. The latter results enable a choice to be made between the two curves given for the film torque in [2]. The paper concludes with some suggestions for extensions of the work.

2. Formulation of the problem

The configuration envisaged is as follows. A circular disk is inserted to zero depth into an infinite plane film-covered fluid interface and is rotated about a vertical axis through its centre with uniform angular speed. Units are chosen so that the disk radius and rotation speed are both unity, and the fluid motion is assumed to be sufficiently slow to permit the linearisation of the Navier–Stokes equations. Cylindrical polar coordinates (ρ, x, ϕ) are taken so that the fluid (assumed infinitely deep) occupies the region $x \geq 0$, the x-axis being coincident with the axis of rotation of the disk. The only non-vanishing component of fluid velocity is then the azimuthal component $w(x, \rho)$ which satisfies, in $x \geq 0$, the equation

$$\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} - \frac{w}{\rho^2} + \frac{\partial^2 w}{\partial x^2} = 0. \tag{1}$$

The boundary conditions on $x = 0$ are

$$w = \rho, \quad 0 \leq \rho \leq 1, \tag{2}$$

and

$$\mu\frac{\partial w}{\partial x} - \eta\frac{\partial^2 w}{\partial x^2} = 0, \quad \rho > 1, \tag{3}$$

where μ and η are respectively the internal and shear viscosities of the substrate and adsorbed surface layer of fluid. Further, both w and $\partial w/\partial x$ must vanish as $(x^2 + \rho^2)^{\frac{1}{2}} \to \infty$.

We next write

$$w(x, \rho) = \rho V^{(3)}(x, \rho);$$

then in $x \geq 0$, $V^{(3)}(x, \rho)$ satisfies the equation

$$L_3 V^{(3)}(x, \rho) = 0,$$

where L_k is the differential operator

$$\frac{\partial^2}{\partial \rho^2} + \frac{k}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x^2}.$$

In the language of GASPT, $V^{(3)}$ is a five-dimensional harmonic (see Appendix 2), and from (2), (3) satisfies on $x = 0$ the conditions

$$V^{(3)}(0, \rho) = 1, \quad 0 \leq \rho \leq 1, \tag{4}$$

$$\mu \frac{\partial V^{(3)}}{\partial x} - \eta \frac{\partial^2 V^{(3)}}{\partial x^2} = 0, \quad \rho > 1. \tag{5}$$

Using equations (A12) and (A13) with $k = 3$, $V^{(3)}(x, \rho)$ can be represented as

$$V^{(3)}(x, \rho) = \frac{1}{\rho^2} \int_0^\rho \frac{y^2 V^{(2)}(x, y)}{(\rho^2 - y^2)^{\frac{1}{2}}} dy \tag{6}$$

$$= - \int_\rho^\infty \frac{y^{-2} V^{(-2)}(x, y)}{(y^2 - \rho^2)^{\frac{1}{2}}} dy, \tag{7}$$

where $V^{(\pm 2)}(x, y)$ are a pair of conjugate four-dimensional harmonics, satisfying the differential equations

$$L_{\pm 2} V^{(\pm 2)}(x, y) = 0,$$

and the Stokes–Beltrami relations (A6):

$$\frac{\partial V^{(2)}}{\partial x} = \frac{1}{y^2} \frac{\partial V^{(-2)}}{\partial y}, \quad \frac{\partial V^{(2)}}{\partial y} = - \frac{1}{y^2} \frac{\partial V^{(-2)}}{\partial x}.$$

Consider next the boundary conditions to be imposed on $V^{(2)}(x, y)$. From (2) and (6),

$$\frac{1}{\rho^2} \int_0^\rho \frac{y^2 V^{(2)}(0, y)}{(\rho^2 - y^2)^{\frac{1}{2}}} dy = 1, \quad 0 \leq \rho \leq 1,$$

an Abel integral equation with solution

$$V^{(2)}(0, y) = 4/\pi, \quad 0 \leq y \leq 1. \tag{8}$$

Similarly, equations (3) and (7) give

$$\eta \frac{\partial^2 V^{(-2)}(0, y)}{\partial x^2} - \mu \frac{\partial V^{(-2)}(0, y)}{\partial x} = 0, \quad y > 1. \tag{9}$$

Using the second of the Stokes–Beltrami relations, (9) may be written as

$$\frac{\partial}{\partial y} \left\{ \eta \frac{\partial V^{(2)}(0, y)}{\partial x} - \mu V^{(2)}(0, y) \right\} = 0, \quad y > 1. \tag{10}$$

To ensure the vanishing of w and $\partial w/\partial x$ as $(\rho^2 + x^2)^{\frac{1}{2}} \to \infty$, we must demand that $V^{(2)}(x, y)$ and $\partial V^{(2)}(x, y)/\partial x \to 0$ as $(x^2 + y^2)^{\frac{1}{2}} \to \infty$. An integration of (10) now shows that

$$\eta \frac{\partial V^{(2)}}{\partial x}(0, y) - \mu V^{(2)}(0, y) = 0, \quad y > 1, \tag{11}$$

a conventional impedance-type boundary condition. The mixed boundary-value problem for $V^{(2)}(x, y)$ may now be specified as that of finding a solution of $L_2 V^{(2)}(x, y) = 0$ in $x \geq 0$, vanishing suitably at infinity, and satisfying the mixed conditions (8) and (11) on $x = 0$.

To solve the problem for the harmonic $V^{(2)}(x, y)$ we first construct a representation using the conventional Green's function approach. Let $G(x, y; x', y')$ satisfy the equation

$$L_2 G(x, y; x', y') = \frac{\pi}{y^2} \delta(x - x')\delta(y - y') \tag{12}$$

in $[0, \infty) \times [0, \infty)$, and the boundary condition

$$\eta \frac{\partial G}{\partial x} - \mu G = 0 \text{ on } x = 0, \quad 0 \leq y < \infty. \tag{13}$$

According to Weinstein [12], a fundamental solution of (12) which is $O(r^{-\frac{3}{2}})$ as $r = (x^2 + y^2)^{\frac{1}{2}} \to \infty$ is

$$G_0(x, y; x', y') = \frac{1}{yy'} \int_0^\infty t^{-1} e^{-|x-x'|t} \sin yt \sin y't\, dt. \tag{14}$$

The appropriate Green's function satisfying (13) now follows as

$$G(x, y; x', y') = \frac{1}{yy'} \int_0^\infty t^{-1} \sin yt \sin y't \times \left\{ e^{-|x-x'|t} + \frac{\lambda t - 1}{\lambda t + 1} e^{-(x+x')t} \right\} dt, \tag{15}$$

where (x, y), $(x', y') \in [0, \infty) \times [0, \infty)$, and an application of Green's theorem (see [12]) yields the representation

$$V^{(2)}(x, y) = \frac{1}{\pi} \int_0^1 y'^2 G(x, y; 0, y') \left\{ \lambda^{-1} V^{(2)}(0, y') - \frac{\partial V^{(2)}}{\partial x'}(0, y') \right\} dy',$$

or, using (8),

$$V^{(2)}(x, y) = \frac{1}{\pi} \int_0^1 y'^2 G(x, y; 0, y') f(y') dy', \tag{16}$$

where

$$f(y) = \frac{4}{\pi\lambda} - \frac{\partial V^{(2)}}{\partial x}(0, y). \tag{17}$$

Again invoking (8), the unknown source distribution $f(y)$ is determined by the integral equation

$$\int_0^1 y'^2 G(0, y; 0, y') f(y') dy' = 4, \quad 0 \leq y < 1. \tag{18}$$

Equation (18) is a Fredholm integral equation of the first kind; in the limit $\lambda \to \infty$, i.e. a very viscous surface layer, its solution is given by

$$f(y) = 4/\pi(1 - y^2)^{\frac{1}{2}}, \tag{19}$$

exhibiting an inverse square root singularity as $y \to 1 - 0$. We therefore transform it into a Fredholm equation of the second kind for a derived function which is regular. To this end, write

$$G(0, y; 0, y') = K_0(y, y') + K_1(y, y'),$$

where

$$K_0(y, y') = \frac{2}{yy'} \int_0^\infty t^{-1} \sin yt \sin y't \, dt,$$

and

$$K_1(y, y') = -\frac{2}{yy'} \int_0^\infty \frac{1}{t(1 + \lambda t)} \sin yt \sin y't \, dt.$$

Equation (18) is equivalent to

$$\int_0^1 y'^2 K_0(y, y') f(y') dy' = 4 - \int_0^1 y'^2 K_1(y, y') f(y') dy', \tag{20}$$

with ([13], page 70)

$$K_0(y, y') = \frac{2}{yy'} \int_0^{\min(y,y')} \frac{t}{(y^2 - t^2)^{\frac{1}{2}}(y'^2 - t^2)^{\frac{1}{2}}} dt.$$

Using Williams' method [10], equation (20) can be transformed into an equation of the second kind, namely

$$T(u) = 1 + \int_0^1 vT(v) \left\{ \int_0^\infty \frac{t}{1 + \lambda t} J_0(ut) J_0(vt) dt \right\} dv, \quad 0 \le u \le 1, \tag{21}$$

where

$$T(u) = \tfrac{1}{2} \int_0^1 \frac{yf(y)}{(y^2 - u^2)^{\frac{1}{2}}} dy. \tag{22}$$

(It is of interest to note that equation (21) has also arisen in work of Chakrabati [14].) It follows from (22) that

$$f(y) = -\frac{4}{\pi y} \frac{d}{dy} \int_y^1 \frac{uT(u)}{(u^2 - y^2)^{\frac{1}{2}}} du; \tag{23}$$

in the special case $\lambda \to \infty$, (21) has the solution $T(u) = 1$, $0 \le u \le 1$, and substitution in (23) confirms the result (19).

We next derive expressions for the substrate and film torques. The substrate torque M exerted on the base of the disk by the fluid is given by

$$\frac{M}{\mu} = 2\pi \int_0^1 \rho^2 \frac{\partial w(0, \rho)}{\partial x} d\rho,$$
$$= 2\pi \int_0^1 \rho^3 \frac{\partial V^{(3)}(0, \rho)}{\partial x} d\rho, \tag{24}$$

where, from (6),

$$\frac{\partial V^{(3)}(0, \rho)}{\partial x} = \frac{1}{\rho^2} \int_0^\rho \frac{y^2}{(y^2 - \rho^2)^{\frac{1}{2}}} \frac{\partial V^{(2)}(0, y)}{\partial x} dy. \tag{25}$$

Substituting (25) into (24), reversing the orders of integration and introducing $f(y)$ defined by (17) then leads to

$$\frac{M}{\mu} = \frac{\pi}{2\lambda} - 2\pi \int_0^1 y^2(1 - y^2)^{\frac{1}{2}} f(y) dy. \tag{26}$$

Equation (26) is now expressed in terms of $T(u)$ by substituting from (23) for $f(y)$; on integration by parts followed by an interchange of orders of integration, the final result is

$$\frac{M}{\mu} = \frac{\pi}{2\lambda} - 8 \int_0^1 uT(u)\{2E(u) - K(u)\} du, \tag{27}$$

where $K(u)$ and $E(u)$ are the complete elliptic integrals of modulus u of the first and second kinds. Since $K(u)$ is singular as $u \to 1$, an alternative form of (27) proves useful in numerical work, namely

$$\frac{M}{\mu} = \frac{\pi}{2\lambda} - \tfrac{8}{3}T(1) + 8 \int_0^1 u^2 T'(u) P(u) du, \tag{28}$$

where

$$P(u) = \int_0^{\frac{1}{2}\pi} \sin^2\theta (1 - u^2 \sin^2\theta)^{\frac{1}{2}} d\theta.$$

The film torque N exerted by the surface layer of fluid on the rim of the rotating disk is given by

$$\frac{N}{\mu} = 2\pi\lambda \lim_{\rho \to 1+0} \frac{\partial}{\partial \rho} \left\{ \frac{w(0, \rho)}{\rho} \right\}$$
$$= 2\pi\lambda \lim_{\rho \to 1+0} \frac{\partial V^{(3)}(0, \rho)}{\partial \rho}, \tag{29}$$

where, from (7),

$$V^{(3)}(0, \rho) = -\int_\rho^\infty \frac{y^{-2} V^{(-2)}(0, y)}{(y^2 - \rho^2)^{\frac{1}{2}}} dy.$$

Carrying out the ρ-differentiation, using the first of the Stokes–Beltrami relations, and the boundary condition (11) on $V^{(2)}(0, y)$ for $y > 1$, it is found that

$$\lim_{\rho \to 1+0} \frac{\partial V^{(3)}}{\partial \rho}(0, \rho) = -2 - \frac{1}{\lambda} \int_1^\infty \frac{y}{(y^2-1)^{\frac{1}{2}}} V^{(2)}(0, y) dy. \tag{30}$$

To express (30) in terms of $T(u)$, we have from (16) and (23) that

$$\begin{aligned} V^{(2)}(0, y) &= -\frac{4}{\pi^2} \int_0^1 y' G(0, y; 0, y') \left\{ \frac{d}{dy'} \int_{y'}^1 \frac{uT(u)}{(u^2-y'^2)^{\frac{1}{2}}} du \right\} dy' \\ &= \frac{4\lambda}{\pi y} \int_0^1 uT(u) \left\{ \int_0^\infty \frac{t}{1+\lambda t} J_0(ut) \sin yt\, dt \right\} du. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^1 \frac{y}{(y^2-1)^{\frac{1}{2}}} V^{(2)}(0, y) dy \\ &= \frac{4\lambda}{\pi} \int_0^1 uT(u) \left\{ \int_0^\infty \frac{t}{1+\lambda t} J_0(ut) \left(\int_1^\infty \frac{\sin yt}{(y^2-1)^{\frac{1}{2}}} dy \right) dt \right\} du \\ &= 2\lambda \int_0^1 uT(u) \left\{ \int_0^\infty \frac{t}{1+\lambda t} J_0(ut) J_0(t) dt \right\} du \\ &= 2\lambda \{T(1) - 1\}, \end{aligned}$$

from the integral equation (21). It follows that

$$\lim_{\rho \to 1+0} \frac{\partial V^{(3)}(0, \rho)}{\partial \rho} = -2 - 2\{T(1) - 1\},$$

and

$$\frac{N}{\mu} = -4\pi\lambda T(1). \tag{31}$$

3. Numerical solution

It can be shown that the kernel of integral equation (21) is logarithmically singular when $u = v$. Using arguments similar to those of Hutson [15], it can be proved that the equation possesses a unique solution for all $\lambda > 0$. However, it does not seem to be possible to solve (21) in closed form; hence either numerical or asymptotic procedures must be used. In this section a numerical solution of (21) is described, based on Mangulis' treatment [11] of an equation of similar structure.

Following Mangulis, the various terms in (21) are expanded in Fourier–Dini series; thus we write

$$T(u) = \sum_{m=0}^{\infty} c_m J_0(\nu_m u)/J_0(\nu_m), \tag{32}$$

where $\nu_0 = 0$, and ν_m is the m-th positive root in ascending order of magnitude of the equation $J_1(\nu) = 0$. Further,

$$J_0(vt) = 2t \sum_{m=0}^{\infty} \frac{1}{(t^2 - \nu_m^2)J_0(\nu_m)} J_1(t)J_0(\nu_m u), \tag{33}$$

and $\int_0^1 vT(v)J_0(vt)dv$ can be evaluated from (32) and (33) using the orthogonality relations

$$\int_0^1 vJ_0(\nu_m v)J_0(\nu_n v)dv = \tfrac{1}{2}\{J_0(\nu_m)\}^2\delta_{mn}. \tag{34}$$

Substituting the various Fourier–Dini expansions in (21) and equating coefficients of $J_0(\nu_m u)$ results in the infinite set of linear equations

$$c_m - \sum_{n=0}^{\infty} h_{mn}c_n = \delta_{m0}, \qquad m = 0, 1, 2, \ldots, \tag{35}$$

where

$$h_{mn} = 2\int_0^{\infty} \frac{t^3}{(1 + \lambda t)(t^2 - \nu_m^2)(t^2 - \nu_n^2)} \{J_1(t)\}^2 dt. \tag{36}$$

Equations (35) can now be solved and $T(u)$ computed to the required degree of accuracy by the method of truncation to a finite set.

Consider next the film and substrate torques. From (31) and (32), we have that

$$\frac{N}{\mu} = -4\pi\lambda \sum_{m=0}^{\infty} c_m. \tag{37}$$

A suitable expression for the substrate torque can be found from (28) by substituting for $T'(u)$ and integrating term-by-term; the result is that

$$\frac{M}{\mu} = \frac{\pi}{2\lambda} - 8 \sum_{m=0}^{\infty} l_m c_m, \tag{38}$$

where the constants l_m are defined by

$$l_m = \tfrac{1}{3} + \frac{\nu_m}{J_0(\nu_m)} \int_0^1 u^2 J_1(\nu_m u)P(u)du. \tag{39}$$

The coefficients l_m are independent of λ and have been calculated numerically.

The numerical solution of equations (35) truncated to a system of $\mathcal{N}$ equations, has been carried out numerically for various values of λ and for $\mathcal{N}$ up to 81. Standard NAG library algorithms for numerical integration and the solution of sets of linear simultaneous were used. The form (36) is not suitable for the numerical calculation of the coefficients h_{mn}; a more convenient expression results from an application of Noble's contour-integral technique [16], namely

$$h_{mn} = -\frac{\pi \nu_n J_0(\nu_n) Y_0(\nu_n)}{2(\lambda \nu_n + 1)} \delta_{mn} - \frac{4\lambda}{\pi} \int_0^\infty \frac{t^4 I_1(t) K_1(t)}{(1 + \lambda^2 t^2)(t^2 + \nu_m^2)(t^2 + \nu_n^2)} dt. \tag{40}$$

Since $I_1(t)K_1(t) \sim (2t)^{-1}$ as $t \to \infty$, convergence at the upper limit for $n, m > 0$ is enhanced by writing

$$tI_1(t)K_1(t) = \{tI_1(t)K_1(t) - \tfrac{1}{2}\} + \tfrac{1}{2},$$

and evaluating in closed form the infinite integral

$$\int_0^\infty \frac{t^3\,dt}{(1 + \lambda^2 t^2)(t^2 + \nu_n^2)(t^2 + \nu_m^2)}.$$

Table 1 shows some sample numerical torque results for various values of λ in the range [.02, 100].

TABLE 1
Values of M/μ and N/μ for various λ.

λ	.02	.025	.05	.075	
$-M/\mu$	4.40	4.33	4.10	3.95	
λ	.1	.2	1	10	100
$-M/\mu$	3.84	3.581	3.078	2.743	2.767
λ	.02	.025	.05	.075	
$-N/\mu$	1.7*	1.8*	2.6*	3.2*	
λ	.1	.2	1	10	100
$-N/\mu$	3.7*	5.62	17.41	132.6	1264.6

Experience showed that substrate torque convergence was satisfactory, and the figures given are believed to be correct to the number of decimal places quoted. Convergence of the film torque series (37) is less satisfactory and doubtful values are marked with an asterisk; thus when $\lambda = .1$, the quoted result 3.7 is of little value. However if the process of series completion* (see [17]) is applied to the first 80 partial sums when $x = 1$ of $\sum_{m=0}^{\infty} d_m x^m$, where $d_m = 4\pi\lambda c_m$, a value of 3.56 is obtained, which compares very favourably with the value estimated from curve B in Fig. 4 of [2]. Similarly, if series completion is applied when $\lambda = .02$, a value of 1.43 is obtained for the film torque, again in agreement with curve B in [2]. Converge of the film-torque with respect to $\mathcal{N}$ before and after completion of the series is illustrated in Table 2.

* The series was completed after $\mathcal{N}$ terms by a remainder proportional to

$$(1 - x)^{\frac{2}{3}} - \sum_{n=0}^{\mathcal{N}-1} \frac{(n - \frac{5}{3})!}{n!\,(-\frac{5}{3})!} x^n.$$

TABLE 2
Film-torque convergence and series completion.

$\mathcal{N}$	$\lambda = .1$		$\lambda = .02$	
	$\sum_{m=0}^{\mathcal{N}-1} d_m$	series completion	$\sum_{m=0}^{\mathcal{N}-1} d_m$	series completion
75	3.7486	3.560	1.6972	1.4322
76	3.7469	3.560	1.6949	1.4322
77	3.7453	3.560	1.6926	1.4324
78	3.7437	3.560	1.6903	1.4324
79	3.7421	3.561	1.6881	1.4325
80	3.7406	3.561	1.6860	1.4325

A comparison of the values for the substrate torque given in Table 1 with those taken from Fig. 3 of [2] shows that for small λ our computed values are greater than those of Goodrich and Chatterjee. Furthermore, our values extrapolate smoothly to those predicted by the asymptotic expression as $\lambda \to 0$, derived in the next section. Thus it appears that the phenomenon of negative excess torque for small λ conjectured in [2] may result from inaccuracies in computation of the substrate torque.

4. Asymptotic analysis

4.1. *The case* $\lambda \gg 1$.
Setting $\sigma = \lambda^{-1}$, equation (21) can be written as

$$T(u) = 1 + \int_0^1 vK(u, v)\, T(v)\, dv,$$

where

$$K(u, v) = \sigma \int_0^\infty \frac{t}{t+\sigma} J_0(ut) J_0(vt)\, dt$$

$$= \sigma \int_0^\infty J_0(ut) J_0(vt)\, dt - \sigma^2 \int_0^\infty \frac{1}{t+\sigma} J_0(ut) J_0(vt)\, dt. \tag{41}$$

In order to expand the second integral in (41) asymptotically for small σ, we re-write it in the form

$$\int_0^\infty e^{-\eta\sigma} \left\{ \int_0^\infty e^{-\eta t} J_0(ut) J_0(vt)\, dt \right\} d\eta = \frac{1}{2\pi} \int_0^\infty e^{-\eta\sigma} \left\{ \int_0^{2\pi} \frac{d\psi}{(\mu^2 + \eta^2)^{\frac{1}{2}}} \right\} d\eta, \tag{42}$$

where

$$\mu^2 = u^2 + v^2 - 2uv \cos \psi.$$

Interchanging orders of integration in (42), and using the result

$$\int_0^\infty \frac{e^{-\eta\sigma}}{(\mu^2+\sigma^2)^{\frac{1}{2}}}\,d\eta = \tfrac{1}{2}\pi\{\boldsymbol{H}_0(\sigma\mu) - Y_0(\sigma\mu)\},$$

where $\boldsymbol{H}_0(x)$ is the Struve function of zero order (see [18], p. 328), it follows that

$$K(u, v) = \sigma \int_0^\infty J_0(ut)J_0(vt)dt - \tfrac{1}{4}\sigma^2 \int_0^{2\pi} \{\boldsymbol{H}_0(\sigma\mu) - Y_0(\sigma\mu)\}\,d\psi.$$

Using known expansions for $\boldsymbol{H}_0(x)$ and $Y_0(x)$ for small values of x, $K(u, v)$ can now be expanded asymptotically in the limit $\sigma \to 0$. To lowest order,

$$\boldsymbol{H}_0(\sigma\mu) - Y_0(\sigma\mu) = -\frac{2}{\pi}\log\sigma + O(1) \text{ as } \sigma \to 0,$$

whence

$$K(u, v) = \sigma \int_0^\infty J_0(ut)J_0(vt)dt + \sigma^2\log\sigma + O(\sigma^2), \text{ as } \sigma \to 0. \tag{43}$$

Using expansion (43), the equation (21) can be solved iteratively, with the result that

$$\begin{aligned} T(u) &= 1 + \frac{2\sigma}{\pi}E(u) + \tfrac{1}{2}\sigma^2\log\sigma + O(\sigma^2), \qquad \sigma \to 0, \\ &= 1 + \frac{2}{\lambda\pi}E(u) - \frac{1}{2\lambda^2}\log\lambda + O(\lambda^{-2}), \qquad \lambda \to \infty. \end{aligned} \tag{44}$$

From (31) and (44) the film torque is found as

$$\frac{N}{\mu} = -4\pi\lambda - 8 + \frac{2\pi}{\lambda}\log\lambda + O(\lambda^{-1}), \qquad \lambda \to \infty. \tag{45}$$

Similarly, substituting (44) into (27), and using the result that

$$\int_0^1 uE(u)\{2E(u) - K(u)\}\,du = \tfrac{1}{2}, \tag{46}$$

the substrate torque has the asymptotic development

$$\frac{M}{\mu} = -\tfrac{8}{3} + \frac{1}{\lambda}\left(\frac{\pi}{2} - \frac{8}{\pi}\right) + \frac{4}{3\lambda^2}\log\lambda + O(\lambda^{-2}), \qquad \lambda \to \infty. \tag{47}$$

When $\lambda = 10$, equations (45) and (47) give torque values of -132.2 and -2.734, compared with the computed values of -132.62 and -2.743. When $\lambda = 100$, (45) and (47) give -1264.3 and -2.676, compared with the computed values of -1264.6 and -2.6759, a very satisfactory agreement which confirms the numerical analysis of Section 3.

4.2. *The case $\lambda \ll 1$.*
To treat the case $\lambda \ll 1$ it is convenient to return to the representation (16) of $V^{(2)}(x, y)$; using the definition (16) of the Green's function G, we have that

$$V^{(2)}(x, y) = \frac{2}{\pi y} \int_0^\infty \frac{e^{-xt}}{1 + \lambda t} \sin yt \left\{ \int_0^1 y' \sin y't(\lambda f(y'))dy' \right\} dt, \tag{48}$$

where

$$\lambda f(y) = \frac{4}{\pi} - \lambda \frac{\partial V^{(2)}}{\partial x}(0, y), \quad 0 \leq y < 1. \tag{49}$$

If $\lambda \ll 1$, as a first approximation we neglect the term $\lambda \partial V^{(2)}(0, y)/\partial x$ in (49) in comparison with $4/\pi$ and set $\lambda f(y) = 4/\pi$. It may be verified that the resulting approximate solution,

$$V^{(2)}(x, y) \simeq \frac{4\sqrt{2}}{\pi^{\frac{3}{2}} y} \int_0^\infty \frac{e^{-xt}}{t^{\frac{1}{2}}(1 + \lambda t)} \sin ty J_{\frac{3}{2}}(t) dt, \tag{50}$$

satisfies the boundary condition $V^{(2)}(0, y) = 4/\pi$, $0 \leq y < 1$, with an error which is $O(\lambda)$ everywhere except in the vicinity of the edge $y = 1$. Thus for $0 < \lambda \ll 1$, we have that

$$V^{(2)}(0, y) \simeq \frac{4\sqrt{2}}{\pi^{\frac{3}{2}} y} \int_0^\infty \frac{1}{t^{\frac{1}{2}}(1 + \lambda t)} \sin yt J_{\frac{3}{2}}(t) dt, \tag{51}$$

and

$$\frac{\partial V^{(2)}(0, y)}{\partial x} \simeq -\frac{4\sqrt{2}}{\pi^{\frac{3}{2}} y} \int_0^\infty \frac{t^{\frac{1}{2}}}{1 + \lambda t} \sin yt J_{\frac{3}{2}}(t) dt. \tag{52}$$

To calculate the substrate torque we have from (24) and (25) that

$$\begin{aligned} -\frac{M}{\mu} &= 2\pi \int_0^1 y^2(1 - y^2)^{\frac{1}{2}} \frac{\partial V^{(2)}}{\partial x}(0, y) dy \\ &\simeq 4\lambda\sqrt{2} \int_0^\infty \frac{1}{t^{\frac{1}{2}}(1 + \lambda t)} J_{\frac{3}{2}}(t) J_2(t) dt \end{aligned} \tag{53}$$

on using (52) and a case of Sonine's first finite integral ([18], p. 373). The infinite integral in (53) can now be expanded asymptotically for small λ using the Mellin transform technique described in [19], with the result that

$$-\frac{M}{\mu} = \tfrac{16}{3} - 4\pi^{\frac{1}{2}}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}}), \quad \lambda \to 0. \tag{54}$$

To calculate the leading term in the film torque as $\lambda \to 0$, (51) is substituted into the formula

$$-\frac{N}{\mu} = 4\pi\lambda + 2\pi \int_0^\infty \frac{y}{(y^2 - 1)^{\frac{1}{2}}} V^{(2)}(0, y) dy.$$

Carrying out the y-integration using Sonine's infinite integral ([13], p. 31) gives the result

$$
\begin{aligned}
-\frac{N}{\mu} &\simeq 4\pi\sqrt{2}\int_0^\infty \frac{1}{t^{\frac{1}{2}}(1+\lambda t)} J_{\frac{1}{2}}(t)J_0(t)dt + O(\lambda)\\
&= 4\pi^{\frac{1}{2}}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}}), \qquad \lambda \to 0.
\end{aligned}
\tag{55}
$$

Results (54) and (55) are the required asymptotic limits as $\lambda \to 0$. Note the $\frac{16}{3} - M/\mu - N/\mu = o(\lambda^{\frac{1}{2}})$, which emphasises the conclusion in [2] that if the surface shear viscosity of the adsorbed layer is small in comparison with the substrate viscosity, the rotating-disk apparatus is not a reliable experimental method.

It has been remarked earlier that for small λ the substrate torque results in [2] are consistently less than our figures. The asymptotic formula (54) gives results for $\lambda \leq .01$ which join on smoothly to numerical results for larger λ; in Fig. 1 we plot our substrate results for $0 \leq \lambda \leq .2$ and include for comparison some points taken from Fig. 3 of [2]. Turning to the film torque, results calculated from (55) for $\lambda \leq .01$ lie on a reasonable extrapolation to the origin of curve B in Fig. 4 of [2]. Thus this evidence, in conjunction with our numerical values for larger λ, confirms curve B as the correct curve.

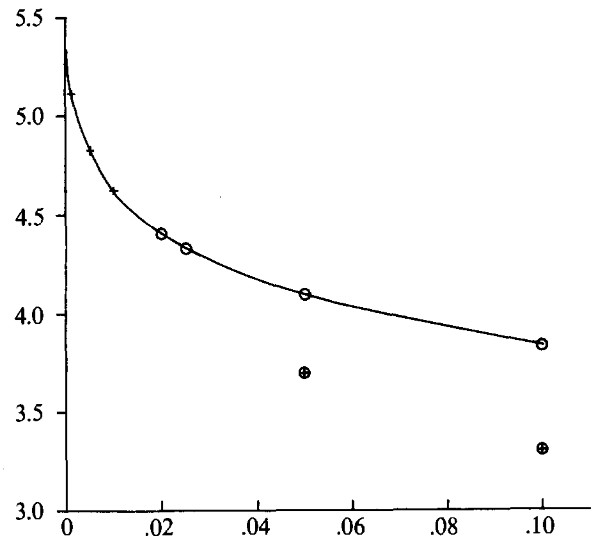

Figure 1. Graph of $-M/\mu$ against λ; + values calculated from the asymptotic formula, ⊙ values calculated numerically, ⊕ values taken from Fig. 3 of [2].

5. Concluding remarks

In this paper we have given a treatment of the rotating disk problem which differs considerably from the original work of Goodrich. There are alternative ways of deriving the governing integral equation (21), and of course the methods used in the paper reflect the personal preferences of the author. The treatment of the problem is complete in that numerical or asymptotic methods have been provided which cover the full range of values of the parameter λ. The one unsatisfactory detail is that I have been unable to give an accurate order estimate of the error in the torque formulae (54) and (55) for small λ. A rigorous derivation would require a detailed consideration of the edge problem, and the difficulty of such problems is well-known (see, e.g. the considerable literature on the two-disk capacitor at small disk separation [20, 21]). However, it is felt that (54) and (55) give accurately the torques correct to $O(\lambda^{\frac{1}{2}})$, and numerical evidence reinforces this conclusion.

The methods of this paper can be used for problems with more complicated geometries, for example fluids in bounded containers. The modification necessary is to replace the Green's function (15) by one appropriate to the image in (x, y)-space of the physical container. The reduction of the problem to that of solving a Fredholm equation of the second kind can then be effected, and numerical and asymptotic solutions derived. The problem of the rotating annulus can also be formulated and solved, since methods exist for treating three-part mixed-boundary value problems (see, e.g. [13]).

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Appendix 1. The limiting case $\mu \to 0$, $\lambda \to \infty$

In this case the velocity component $w(x, \rho)$ satisfies (1) and the conditions

$$w(0, \rho) = \rho, \qquad 0 \leq \rho \leq 1, \tag{A1}$$

$$\frac{\partial^2 w(0, \rho)}{\partial x^2} = 0, \qquad \rho > 1. \tag{A2}$$

It follows from (1) and (A2) that

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\right) w(0, \rho) = 0, \qquad \rho > 1,$$

which has the general solution

$$w(0, \rho) = A\rho + B/\rho, \qquad \rho > 1.$$

The arbitrary constants A, B are found as 0 and 1 from the requirement that $w \to 0$ as $\rho \to \infty$ and the continuity of $w(0, \rho)$ at $\rho = 1$. Thus

$$w(0, \rho) = \begin{cases} \rho, & 0 \leq \rho \leq 1, \\ \dfrac{1}{\rho}, & \rho > 1, \end{cases} \tag{A3}$$

in agreement with [1]. The film torque N is found from (A3) as $-4\pi\eta$, confirming the appropriate limit of (46). The velocity field in the fluid may be expressed as a Hankel transform as in [1]; however this transform may not be legitimately differentiated twice with respect to x and x set equal to zero.

Appendix 2. Complementary representations and GASPT

Let (ρ, x, ϕ) be cylindrical polar coordinates, and define the differential operator L_k as

$$L_k = \frac{\partial^2}{\partial \rho^2} + \frac{k}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial x^2}, \tag{A4}$$

where k is a real number. A solution of the differential equation

$$L_k v = 0 \tag{A5}$$

which is finite on the axis $\rho = 0$ is denoted by $V^{(k)}(x, \rho)$; if $k > 0$, then in the terminology of GASPT, $V^{(k)}$ is an axially symmetric harmonic function in a space of $k + 2$ dimensions. We associate with $V^{(k)}$ a conjugate harmonic $V^{(-k)}$, where

$$L_{-k} V^{(-k)} = 0$$

and the conjugate pair $(V^{(k)}, V^{(-k)})$ satisfies the Stokes–Beltrami relations

$$\frac{\partial V^{(k)}}{\partial x} = \frac{1}{\rho^k}\frac{\partial V^{(-k)}}{\partial \rho}, \quad \frac{\partial V^{(k)}}{\partial \rho} = -\frac{1}{\rho^k}\frac{\partial V^{(-k)}}{\partial x}. \tag{A6}$$

A particular pair of harmonics, finite in $x \geq 0$, are

$$V^{(k)} = -\rho^{\frac{1}{2}(-k+1)} J_{\frac{1}{2}(k-1)}(s\rho) e^{-sx}, \tag{A7}$$

$$V^{(-k)} = \rho^{\frac{1}{2}(k+1)} J_{\frac{1}{2}(k+1)}(s\rho) e^{-sx}, \tag{A7}$$

where $s > 0$.

Consider next a space of $k + 1$ dimensions in which the axial and radial coordinates are (x, y). Using (A7) and superposition we can construct the conjugate harmonics

$$\begin{aligned} V^{(k-1)}(x, y) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} y^{-\frac{1}{2}k+1} \int_0^\infty A(s) J_{\frac{1}{2}k-1}(sy) e^{-sx} ds, \\ V^{(-k+1)}(x, y) &= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} y^{\frac{1}{2}k} \int_0^\infty A(s) J_{\frac{1}{2}k}(sy) e^{-sx} ds, \end{aligned} \tag{A8}$$

where $A(s)$ is chosen to ensure convergence of the integrals. The pair $(V^{(k-1)}, V^{(-k+1)})$ vanishes at infinity in the half-space $x > 0$; further $V^{(k-1)}$ is an even function of y and $V^{(-k+1)}(x, 0) = 0$.

By the Sonine integrals, we can write

$$\int_0^\rho (\rho^2 - y^2)^{-\frac{1}{2}} y^{\frac{1}{2}k} J_{\frac{1}{2}k-1}(sy) dy = \left(\frac{\pi}{2s}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}(k-1)} J_{\frac{1}{2}(k-1)}(s\rho), \tag{A9}$$

$$\int_{\rho}^{\infty}(y^2-\rho^2)^{-\frac{1}{2}}y^{-\frac{1}{2}k+1}J_{\frac{1}{2}k}(sy)dy=\left(\frac{\pi}{2s}\right)^{\frac{1}{2}}\rho^{\frac{1}{2}(-k+1)}J_{\frac{1}{2}(k-1)}(s\rho), \tag{A10}$$

where $k \geq 1$. Thus, introducing the $(k+2)$-dimensional harmonic

$$V^{(k)}(x,\rho)=\rho^{\frac{1}{2}(-k+1)}\int_0^{\infty}s^{-\frac{1}{2}}A(s)J_{\frac{1}{2}(k-1)}(s\rho)e^{-sx}ds, \tag{A11}$$

it follows from (A8), (A9), (A10) and (A11) that

$$V^{(k)}(x,y)=\frac{1}{\rho^{k-1}}\int_0^{\rho}\frac{y^{k-1}V^{(k-1)}(x,y)}{(\rho^2-y^2)^{\frac{1}{2}}}dy \tag{A12}$$

$$=-\int_{\rho}^{\infty}\frac{y^{-k+1}V^{(-k+1)}(x,y)}{(y^2-\rho^2)^{\frac{1}{2}}}dy. \tag{A13}$$

(A12) and (A13) are the basic complementary representations. The case $k = 1$ was given originally by Ranger [8]; the conjugate harmonics $V^{(\pm 0)}(x, y)$ then satisfy the Cauchy–Riemann equations and $f(z) = V^{(+0)}(x, y) + iV^{(-0)}(x, y)$ is an analytic function of $z = x + iy$ (this appearance of the symbol z is the motivation behind the non-standard choice of x as the axial cylindrical polar coordinate in this paper).

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